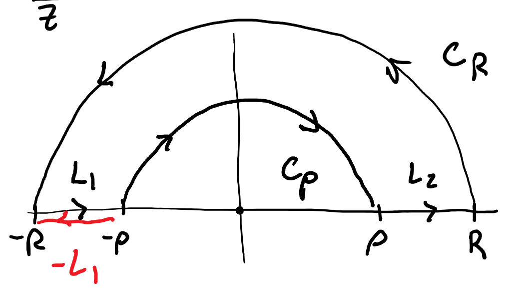
Indented Path

An indented path can sometimes be used to avoid an isolated singularity or branch point that lies on the real axis. We illustrate this method through an example.

Example (Dirichlet's Integral) Compute $\int_0^\infty \frac{\sin x}{x}\,dx$.

We integrate $f(z) = \frac{e^{iz}}{z}$ over the contour



$C = C_R + L_1 + C_\rho + L_2$

By Cauchy-Goursat

$$0 = \int_C \frac{e^{iz}}{z}\,dz = \int_{C_R} \frac{e^{iz}}{z}\,dz + \int_{L_1} \frac{e^{iz}}{z}\,dz + \int_{C_\rho} \frac{e^{iz}}{z}\,dz + \int_{L_2} \frac{e^{iz}}{z}\,dz.$$

Parametrize $-L_1, L_2$ as follows

$-L_1: \quad z(t) = -t, \quad \rho \le t \le R$

$L_2: \quad z(t) = t, \quad \rho \le t \le R.$

Then

$$\int_{L_2} \frac{e^{iz}}{z}\,dz + \int_{L_1} \frac{e^{iz}}{z}\,dz = \int_{L_2} \frac{e^{iz}}{z}\,dz - \int_{-L_1} \frac{e^{iz}}{z}\,dz$$

$$= \int_\rho^R \frac{e^{it}}{t}\,dt - \int_\rho^R \frac{e^{-it}}{t}\,dt$$

$$= \int_\rho^R \frac{e^{it} - e^{-it}}{t}\, dt$$

$$= 2i \int_\rho^R \frac{\sin t}{t}\, dt.$$

Hence,

$$\int_\rho^R \frac{\sin x}{x}\, dx = -\frac{1}{2i}\left(\int_{C_R} \frac{e^{iz}}{z}\, dz + \int_{C_\rho} \frac{e^{iz}}{z}\, dz \right)$$

Thus,

$$\int_0^\infty \frac{\sin x}{x}\, dx = -\frac{1}{2i}\left(\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\, dz + \lim_{\rho\to 0} \int_{C_\rho} \frac{e^{iz}}{z}\, dz \right).$$

By Jordans Lemma, with $M_R := \frac{1}{R}$, $\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z}\, dz = 0$.

To compute $\lim_{\rho\to 0} \int_{C_\rho} \frac{e^{iz}}{z}\, dz$, consider the Laurent series

$$\frac{e^{iz}}{z} = \frac{1}{z} \sum_{n=0}^\infty \frac{(iz)^n}{n!} = \sum_{n=0}^\infty \frac{i^n z^{n-1}}{n!} = \frac{1}{z} + \sum_{n=1}^\infty \frac{i^n z^{n-1}}{n!}.$$

$$= \frac{1}{z} + g(z).$$

Since $g(z)$ is a Taylor series about zero, it is analytic in a neighborhood of zero. Hence, there is a closed disk $|z| \le \varepsilon$ on which $g(z)$ is continuous. Hence, $g(z)$ is bounded near zero, say

$$|g(z)| \le M \quad \text{for all } |z| \le \varepsilon.$$

Hence, if $\rho < \varepsilon$,

$$\left| \int_{C_\rho} g(z)\, dz \right| \overset{\text{T.I.}}{\leq} \pi \cdot \rho \max_{z \in C_\rho} |g(z)|$$

$$\leq \pi \cdot \rho \cdot M \xrightarrow{\rho \to 0} 0 .$$

Thus, $$\lim_{\rho \to 0} \int_{C_\rho} \frac{e^{iz}}{z}\, dz = \lim_{\rho \to 0} \int_{C_\rho} \frac{1}{z}\, dz + \lim_{\rho \to 0} \int_{C_\rho} g(z)\, dz$$

$$= -\pi i + 0$$

Finally, $$\int_0^\infty \frac{\sin x}{x}\, dx = -\frac{1}{2i}\left(-\pi i\right) = \frac{\pi}{2} .$$

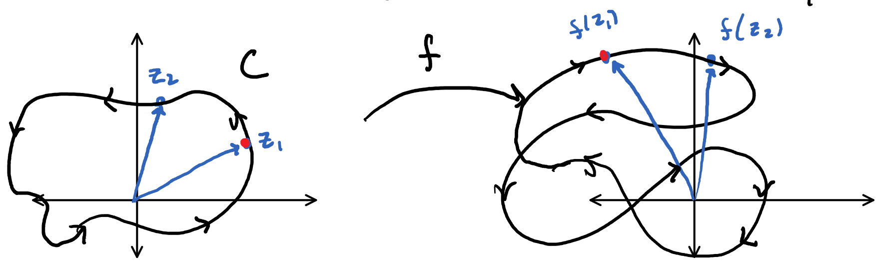
Argument Principle

A function is meromorphic on a domain D if it is analytic on D except for poles.

Let C be a simple closed positively oriented contour and denote by D its interior. Suppose f is meromorphic on D and analytic and nonzero on C. The image

$$f(C) = \{ f(z) : z \in C \}$$

of C under f is also a closed contour, but not necessarily simple.



Since f is nonzero on C, it follows that $f(C)$ does not cross the origin. Consequently, the winding number of $f(C)$ about $z=0$ is defined:

$$n(f(C), 0) = \frac{1}{2\pi i} \int_{f(C)} \frac{1}{z} \, dz .$$

In PSet 5, we saw that the winding number is interpreted geometrically as the number of times the contour winds around a point. We can write

$$n(f(C), 0) = \frac{1}{2\pi} \Delta_C \operatorname{Arg} f(z)$$

where $\Delta_C \operatorname{Arg} f(z)$ is the change in argument of $f(z)$ as C is traversed once in the positive direction.

The argument principle shows that $n(f(C), 0)$ depends only on the number of poles and zeros, counting orders, that lie interior in D.

Theorem (Argument Principle) Let C be a simple closed positively oriented contour and D its interior. Suppose that

(1) f is meromorphic on D;

(2) f is analytic and nonzero on C and not identically zero on D;

(3) Counting orders, Z is the number of zeros of f in D, and P is the number of poles of f in D.

Then
$$n(f(C), 0) = Z - P.$$

Proof. First of all, the number of poles of f in D is finite. Label the poles:

$$p_1, \dots, p_K \quad \text{with orders} \quad m_1, \dots, m_K.$$

Also, the number of zeros is finite since f is not identically zero on D. Label the zeros:

$$z_1, \dots, z_\ell \quad \text{with orders} \quad n_1, \dots, n_\ell.$$

Note that: $Z = \sum_{j=1}^{\ell} n_j$ and $P = \sum_{j=1}^{K} m_j$. Now, consider the function $\frac{f'(z)}{f(z)}$. Note that $\frac{f'(z)}{f(z)}$ is analytic on C and on D except possibly at the zeros or poles of f.

Since f has a zero of order n_j at z_j, we can write
$$f(z) = (z - z_j)^{n_j} g(z)$$
for some $g(z)$ that is analytic and nonzero at z_j.
Then
$$\frac{f'(z)}{f(z)} = \frac{n_j (z-z_j)^{n_j - 1} g(z) + (z-z_j)^{n_j} g'(z)}{(z-z_j)^{n_j} g(z)}$$

$$= \frac{n_j}{z - z_j} + \frac{g'(z)}{g(z)}.$$

Since $\frac{g'(z)}{g(z)}$ is analytic at z_j, this shows that $\frac{f'(z)}{f(z)}$ has a simple pole at z_j with

$$\operatorname*{Res}_{z=z_j} \frac{f'(z)}{f(z)} = n_j .$$

Similarly, at each pole p_j w/ order m_j, we can write

$$f(z) = \frac{\phi(z)}{(z-p_j)^{m_j}} = (z-p_j)^{-m_j} \phi(z)$$

where $\phi(z)$ is analytic and nonzero at p_j. Then

$$\frac{f'(z)}{f(z)} = \frac{-m_j (z-p_j)^{-m_j-1} \phi(z) + (z-p_j)^{-m_j} \phi'(z)}{(z-p_j)^{-m_j} \phi(z)}$$

$$= \frac{-m_j}{z-p_j} + \frac{\phi'(z)}{\phi(z)} .$$

Hence, p_j is a simple pole of $\frac{f'(z)}{f(z)}$ w/ residue

$$\operatorname*{Res}_{z=p_j} \frac{f'(z)}{f(z)} = -m_j .$$

Then we obtain

$$\begin{aligned} n(f(C),0) &= \frac{1}{2\pi i} \int_{f(C)} \frac{1}{z}\, dz \\ &= \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz \\ &= \sum_{j=1}^{l} \operatorname*{Res}_{z=z_j} \frac{f'(z)}{f(z)} + \sum_{j=1}^{K} \operatorname*{Res}_{z=p_j} \frac{f'(z)}{f(z)} \\ &= \sum_{j=1}^{l} n_j - \sum_{j=1}^{K} m_j \\ &= Z - P. \end{aligned}$$

$\boxed{\text{Example}}$ Let C' be a contour parameterized via

$$w(t) = \frac{(2e^{it}-1)^7}{e^{3it}}, \quad 0 \le t \le 2\pi.$$

Compute $n(C', 0)$.

Consider $f(z) = \frac{(2z-1)^7}{z^3}$. Then $f(C) = C'$ where

C is the unit circle $(z(t) = e^{it},\ 0 \le t \le 2\pi)$. By the argument principle,

$$\begin{aligned} n(C', 0) &= n(f(C), 0) \\ &= Z - P. \end{aligned}$$

To compute Z: write $g(z) = \frac{2^7}{z^3}$. Then $f(z) = (z - \frac{1}{2})^7 g(z)$ and $g(z)$ is analytic and nonzero at $z = \frac{1}{2}$. So z is a zero of order 7 of f, and it lies interior to C. So $Z = 7$.

To compute P: define $\phi(z) = (2z-1)^7$. Then $f(z) = \frac{\phi(z)}{z^3}$ and ϕ is analytic and nonzero at $z=0$. So $z=0$ is a pole of order 3 and $z=0$ lies interior to C. So $P = 3$.

Hence,
$$\begin{aligned} n(C', 0) &= 7 - 3 \\ &= 4. \end{aligned}$$

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Rouche's Theorem

Theorem (Rouche) Let C be a simple closed contour and suppose that

(1) $f(z)$ and $g(z)$ are analytic on and interior to C;
(2) $|f(z)| > |g(z)|$ for all $z \in C$.

Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting orders, interior to C.

Proof. By (2), $|f(z)| > |g(z)| \geq 0$ for all $z \in C$, so f has no zeros on C. Moreover, for all $z \in C$,

$$|f(z)+g(z)| \geq \big|\, |f(z)| - |g(z)| \,\big| > 0 .$$

Hence, also $f+g$ has no zeros on C. Denote by Z_f and Z_{f+g} the number of zeros of f and $f+g$, counting multiplicities, that lie interior to C. By the argument principle

$$\begin{aligned}
Z_{f+g} &= n\left((f+g)(C), 0\right) \\
&= \frac{1}{2\pi} \Delta_C \operatorname{Arg}\left(f(z)+g(z)\right) \\
&= \frac{1}{2\pi} \Delta_C \operatorname{Arg}\left(f(z)\left(1+\frac{g(z)}{f(z)}\right)\right) \\
&= \frac{1}{2\pi} \Delta_C \operatorname{Arg}\left(f(z)\right) + \frac{1}{2\pi} \Delta_C \operatorname{Arg}\left(1+\frac{g(z)}{f(z)}\right) \\
&= Z_f + n\left(F(C), 0\right)
\end{aligned}$$

where $F(z) = 1 + \frac{g(z)}{f(z)}$. Now, let $z \in C$. Then

$$|F(z)-1| = \frac{|g(z)|}{|f(z)|} < 1.$$

This proves that $F(C)$ is contained in the open disk $D_1(1)$. Since $0 \notin D_1(1)$, we conclude that $n(F(C), 0) = 0$. This proves

$$Z_f = Z_{f+g}.$$

$\blacksquare$

$\boxed{\text{Example}}$ Determine the number of zeros, counting orders, of

$$p(z) = z^7 - 4z^3 + z - 1$$

that lie interior to the unit circle $|z| = 1$.

The strategy: Choose $f(z)$ and $g(z)$ as in Rouche's theorem, such that

$$p(z) = f(z) + g(z).$$

Moreover, the number of zeros of f interior to C should be easy to compute.

For this problem, define $f(z) = z^7 - 4z^3$ and $g(z) = z - 1$. Note that $f(z) + g(z) = p(z)$ and $f(z)$ has three zeros interior to C. Let $z \in C$ so that $|z| = 1$. Then

$$|f(z)| = |z^7 - 4z^3| = |z|^3 |z^4 - 4|$$
$$\geq 1 \cdot \big| |z|^4 - 4 \big| = |1 - 4| = 3.$$

Also,

$$|g(z)| = |z - 1| \leq |z| + 1 = 2.$$

Hence, $|f(z)| \geq 3 > 2 \geq |g(z)|$. By Rouche's theorem, $p(z)$ has three zeros interior to C.

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$\boxed{\text{Example}}$ Determine the number of zeros, counting orders, of

$$p(z) = 2z^5 - 6z^2 + z + 1$$

that lie in the annulus $1 \le |z| < 2$.

First, compute the number of zeros interior to the circle $C_2(0)$. Take $f(z) = 2z^5 + 1$ and $g(z) = 1 - 6z^2$. If $|z| = 2$, then

$$|f(z)| = |2z^5 + 1| \ge \left| 2|z|^5 - 1 \right| = \left| 2 \cdot 2^5 - 1 \right| = 63.$$

Also,

$$|g(z)| = |1 - 6z^2| \le 1 + 6|z|^2 = 25.$$

Hence, $|f(z)| \ge 63 > 25 \ge |g(z)|$ so by Rouché $p(z)$ has five zeros interior to $C_2(0)$ since $f(z)$ does.

Next, consider the unit circle C. Let $f(z) = -6z^2$ and $g(z) = 2z^5 + z + 1$. Then if $|z| = 1$,

$$|f(z)| = 6|z|^2 = 6,$$

while

$$|g(z)| \le 2|z|^5 + |z| + 1 = 4.$$

So $|f(z)| = 6 > 4 \ge |g(z)|$. By Rouché, $p(z)$ has 2 zeros interior to C since $f(z)$ does. Hence, $p(z)$ has $5 - 2 = 3$ zeros in the annulus. //